# PHYS 798C Spring 2024
# Lecture 28 Summary

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## I. $^3He$ AS A PROTOTYPICAL FERMI LIQUID AND "EXOTIC SUPERCONDUCTOR"

Helium does not solidify at any temperature at atmospheric pressure. It's small mass and large deBroglie wavelength render it a 'quantum fluid'. The weak interactions between He atoms, along with the large zero-point motion of the atoms, allow for condensation into a fluid state at cryogenic temperatures, but not a solid.
$^4He$ is a well-know Bosonic atom (2 protons, 2 neutrons and 2 electrons, all spin-1/2 particles that 'pair up') that undergoes a superfluid transition at about 2.2 K in to a Bose-Einstein-condensate-like state. It's small mass and lack of chemistry makes it ideal for showing Bosonic quantum fluid properties.
$^3He$ on the other hand has one un-paired neutron in the nucleus and a net spin-1/2 nucleus, making it a Fermion. It is even lighter than $^4He$, making for even more interesting quantum effects. At low temperatures it condenses into a fluid which shows prototypical Landau Fermi Liquid properties. Eventually, at about 2.8 mK (three orders of magnitude lower than the superfluid transition temperature of $^4He$), it makes a transition in to a superfluid state that resembles BCS superconductivity, although without the electrical charge.
Our objective is to review the Fermi liquid properties of $^3He$ and then discuss the superfluid state, making contact with BCS theory whenever possible. This superfluid state is similar in many ways to the properties of exotic superconductors, including many heavy Fermion materials.

## II. THE LANDAU FERMI LIQUID THEORY OF $^3He$

(Advice: read sections 7.2-7.4 of James Annett's book "Superconductivity, Superfluids and Condensates"!)
The interatomic potential between He atoms is approximated by the Lennard-Jones (or 6-12) potential: $V(r) = V_0[(\frac{\sigma}{r})^{12} - (\frac{\sigma}{r})^6]$, with $\sigma = 2.6\ \mathring{A}$ and $V_0 \approx 1\ meV \approx 10\ K$. One important feature of this potential is the extremely strong repulsion between He atoms at short distances. This is much stronger than the mutual repulsion between electrons in a metal, and will play an important role in superfluidity in $^3He$.
$^3He$ condenses into a fluid at $T = 3.197\ K$ at atmospheric pressure. At low temperatures (3 to 100 mK) $^3He$ is a degenerate Fermi liquid with a well-defined Fermi surface. The Fermi energy $E_F = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3}$ is equal to just 0.5 meV, as opposed to about 10 eV in good metals. The corresponding Fermi temperature is just $T_F = 4.9$ K, and the Fermi wavenumber is 0.78 $\mathring{A}^{-1}$.

The heat capacity above $T_c$ is observed to be linear in temperature, just like the electronic heat capacity in metals, but with a slope 3 times bigger than theory, $C = \frac{\pi^2 k_B^2 NT}{2E_F}$.

The magnetic properties are dominated by the un-paired nuclear spin, which has a moment $\mu_N = 5.4 \times 10^{-4}\mu_B$, where $\mu_B$ is the Bohr magneton. There is a temperature-independent paramagnetic susceptibility at temperatures below 1 K, analogous to Pauli paramagnetism for the electron gas in a metal.

Landau developed a version of Fermi liquid theory in 1956 to explain the low-temperature properties of $^3He$. Because $^3He$ is a liquid, one can use a spherical Fermi surface and invoke many useful symmetry constriants on the theory. He started with the non-interacting gas and treated the interactions as a perturbation. In the non-interacting case, he treated all particles as de-localized, and used momentum eigenstates, $\psi \sim \frac{1}{\sqrt{V}} e^{-i\vec{k}\cdot\vec{r}}\chi$ similar to BCS. Here $V$ is the volume of the fluid and $\chi$ is the spinor wavefunction. After turning on the interactions it is assumed that there is a 1:1 correspondence between the states of the system, and that the interactions simply change the energies of these states. By *adiabatic continuation*, the wavefunctions smoothly evolve as the interactions are turned on. The Hamiltonian has the form,
$\mathcal{H} = \sum_{i=1}^{N} -\frac{\hbar^2}{2m}\nabla_i^2 + \frac{\lambda}{2}\sum_{i\neq j} V(\vec{r}_i - \vec{r}_j)$. Here $\lambda$ is tuned from 0 (no interactions) to 1 (full interactions)

adiabatically. It is assumed that the atoms interact with each other purely by means of two-body interactions.

Because $^3He$ is an isotropic and degenerate Fermi fluid, the interactions between particles can be written in terms of a small number of parameters as follows. Write the interaction energy as,
$E_{int} = f_1(\vec{k}, \vec{k}') + f_2(\vec{k}, \vec{k}')\vec{S} \cdot \vec{S}'$, making use of the isotropy of the fluid. The first term depends only on the momentum of the atoms, which is typically very close to $k_F$. The second term is the spin-dependent part, which depends on the vector dot product of the two spins, the simplest form of interaction given the spherical symmetry of the fluid. Focus on two atoms on the Fermi sphere, described by $(\vec{k}, \vec{S})$ and $(\vec{k}', \vec{S}')$. Due to the symmetry of the fluid, their interaction energy can only depend on $\vec{k} \cdot \vec{k}' = k_F^2 \cos(\Theta)$ and $\vec{S} \cdot \vec{S}'$. Using the angle $\Theta$ between the directions $\vec{k}$ and $\vec{k}'$, Landau defined his dimensionless parameters $F$ and $G$ as follows,
$F(\Theta) = D(E_F)f_1(\Theta) = \sum_{n=0}^{\infty} F_n P_n(\cos(\Theta)) = F_0 + F_1 \cos\Theta + ...$, and
$G(\Theta) = D(E_F)f_2(\Theta) = \sum_{n=0}^{\infty} G_n P_n(\cos(\Theta)) = G_0 + G_1 \cos\Theta + ...$,
where $D(E_F)$ is the density of states at the Fermi energy, and one has expanded in a set of Legendre polynomials over the spherical Fermi surface. It turns out that most of the normal state physical properties can be understood in terms of just $F_0$, $F_1$ and $G_0$. For example, it is found that the mass of the quasiparticle is enhanced as $m_3^* = m_3(1 + \frac{1}{3}F_1)$, which turns out to be about a factor of 3. This explains the enhanced slope of the heat capacity vs. temperature. It is also found that the magnetic susceptibility is enhanced as $\chi = \frac{m_3^*}{m_3} \frac{\chi^{ideal}}{1+\frac{1}{4}G_0}$ because $G_0 < 0$.

The Landau parameters for $^3He$ as a function of pressure are,

| | $Pressure\ (bar)$ | | |
|---|---|---|---|
| | 1 | 15 | 30 |
| $F_0$ | 10 | 46 | 82 |
| $F_1$ | 6 | 11 | 14.6 |
| $G_0$ | $-2.69$ | $-2.92$ | $-2.95$ |

Note that the solid phase sets in just above a pressure of 30 bar. As the pressure increases, the momentum-dependent $F_0$ and $F_1$ terms grow because the atoms are being forced to interact with each other more strongly. However, note that the spin-dependent terms are very weakly dependent on pressure. The spin is buried in the nucleus and is less sensitive to what is going on 'outside'.

## III. SUPERFLUID PROPERTIES OF $^3He$

$^3He$ shows a heat capacity that is linear in temperature above the superfluid transition. At the transition there is a discontinuity in the heat capacity (see the class web site), similar to that seen in a BCS superconductor.
$^3He$ has three distinct superfluid phases, A, B and A1. In zero magnetic field it will condense in to either the A or B phase, depending on the pressure. With increasing field the B phase is reduced and the A phase takes over. The B phase supports persistent angular momentum states while the A phase does not. In non-zero field there are two second order phase transitions as the system goes from the normal phase to the A1 phase to the A phase. The transition from A to B phase is first order, with latent heat and hysteresis.

The short-range Lennard-Jones ($1/r^{12}$) repulsion is stronger than that in electron-electron interactions in metals, such that the spin-singlet s-wave pairing channel is suppressed. The $\ell = 1$ angular momentum pairing state with spin-triplet pairing is favored. A non-zero angular momentum for the paired atoms helps to keep them away from each other, thus avoiding the short-range repulsion. The large paramagnetic susceptibility of the $^3He$ atoms favors the $S = 1$ pairing. For the pair, the angular momentum vector and the spin vector can, in general, point in different directions, making for many possible pairing states.

## IV. PAIRING IN $^3He$

Leggett (1975) found that the pairing interaction between $^3He$ atoms can be written as,
$V_{k,k'} \approx \frac{1}{D(E_F)} \frac{G_0}{1+\frac{1}{4}G_0} \vec{S} \cdot \vec{S}'$. (This expression is accurate for $|\vec{k} - \vec{k}'| << k_F$) We saw above that $G_0 \approx -3$,

hence the pairing interaction favors ferromagnetic alignment of the atom nuclear spins. Note that there are no phonons in a liquid, hence the phonon-mediated pairing mechanism is not likely to play a role here. The tendancy for ferromagnetic ordering is also a hallmark of the heavy Fermion superconductors that pair in spin-triplet states.

The pairing interaction is now more complicated because we have to keep more careful track of the spins,

$$H_{int} = \sum_{k,k';\alpha\beta\gamma\delta} V_{\alpha\beta\gamma\delta}(k,k')\ c^{+}_{k',\alpha} c^{+}_{-k',\beta} c_{-k,\gamma} c_{k,\delta}.$$

Here $\alpha$, $\beta$, $\gamma$, and $\delta$ label the spin states of the $^3$He atoms, namely either $\uparrow$ or $\downarrow$ since they are spin-1/2 particles. This is a generalization of the BCS pairing Hamiltonian. It is again assumed that Cooper pairs have zero net center of mass momentum in the ground state (i.e. the pairs are made up of $\vec{k}$ and $-\vec{k}$ momentum states).

One can define a BCS-like order parameter as follows,
$F_{\alpha\beta}(k) = \langle c_{-k,\alpha} c_{k,\beta} \rangle$
$= \begin{pmatrix} \langle c_{-k,\uparrow} c_{k,\uparrow} \rangle & \langle c_{-k,\uparrow} c_{k,\downarrow} \rangle \\ \langle c_{-k,\downarrow} c_{k,\uparrow} \rangle & \langle c_{-k,\downarrow} c_{k,\downarrow} \rangle \end{pmatrix}$
giving pairing amplitude in 4 different channels since each nuclear spin can be in one of two states.

The BCS gap equation is now,
$\Delta_{\alpha\beta}(k) = \sum_{k',\gamma\delta} V_{\alpha\beta\gamma\delta}(k,k') \langle c_{-k',\gamma} c_{k',\delta} \rangle$.
The final gap function can be written as,

$$\begin{pmatrix} \Delta_{\uparrow\uparrow}(k) & \Delta_{\uparrow\downarrow}(k) \\ \Delta_{\downarrow\uparrow}(k) & \Delta_{\downarrow\downarrow}(k) \end{pmatrix} = i \left( \Delta_k I_{2\times 2} + \vec{d}(k) \cdot \vec{\sigma} \right) \sigma_y,$$

where $\vec{\sigma}$ is the vector of Pauli spin matrices and $\vec{d}(k)$ is a vector order parameter for the spin triplet pairing state.

The quasiparticle excitation spectrum can be written as,
$E_k = \sqrt{(\epsilon_k - \mu)^2 + |\vec{d}(k)|^2}$, so $|\vec{d}(k)|$ acts like the BCS gap in determining the excitation spectrum. This has important ramifications for persistent angular momentum.

The two most important pairing states in supefluid $^3He$ are,
1) The Anderson-Brinkman-Morel (ABM) or A-phase. In this case, the order parameter has the form, $\vec{d}(k) = (\sqrt{\frac{3}{4\pi}} \sin\theta_k (\cos\phi_k + \sin\phi_k), 0, 0)$, where the angles are the traditional polar angles on the spherical Fermi surface. In this case $|\vec{d}(k)| \sim \sin\theta_k$ and the gap function goes to zero at the north and south poles of the Fermi surface (point nodes). The order parameter points in the $x$-direction all over the Fermi sphere. Due to the point nodes, there are excitations out of the ground state available at arbitrarily low temperatures, hence many properties show power-law-in-temperature behavior rather than the exponentially-activated behavior seen in fully-gapped superfluids/superconductors. There is no persistent angular momentum in the A-phase due to the existence of excitations at arbitrarily small energies near the nodes. The wavefunction is made up of the $S_z = \pm 1$ components of the spin singlet (in other words, the $|\uparrow,\uparrow\rangle$ and $|\downarrow,\downarrow\rangle$ states). The A1 phase (in the presence of a magnetic field) favors one of these two $S_z$ states.

Why does the A-phase of superfluid $^3He$ not support a persistent current? For example a d-wave superconductor has nodes in the energy gap but it still supports quantized vortices and persistent currents. Also superfluid $^4He$ supports persistent currents. Why don't we have this also for the A-phase? The answer is that s-wave and d-wave superconductors, along with $^4He$, are all described by a *scalar* complex order parameter of the form $\psi = |\psi| e^{i\phi}$. As such one can derive fluxoid and circulating current quantization conditions by demanding that the macroscopic quantum wavefunction be single valued. This puts a quantization constraint on the phase winding number. In $^3He$ A-phase and in p-wave (or f-wave) superconductors the order parameter is a vector and is not simply constrained as in the complex order parameter case. In fact, these systems are much richer and can support a variety of exotic topological defects that are far more interesting than vortices!

2) The Balain-Werthamer (BW) or B-phase. In this case, the order parameter has the form, $\vec{d}(k) = \sqrt{\frac{3}{4\pi}}(\sin\theta_k \cos\phi_k, \sin\theta_k \sin\phi_k, \cos\theta_k)$. In this case the order parameter is pointed radially outward on the Fermi surface, and it has a non-zero magnitude everywhere. The system is fully gapped and shows exponentially activated properties at low temperature. The wavefunction is made up of the $S_z = 0$ part of the spin-triplet wavefunction (in other words, the $\frac{|\uparrow,\downarrow\rangle + |\downarrow,\uparrow\rangle}{\sqrt{2}}$ state). This phase supports persistent current states that are more exotic than the vortices that we have studied up to this point.

## V. UNCONVENTIONAL SUPERCONDUCTORS

(Advice: read section 7.5 of James Annett's book "Superconductivity, Superfluids and Condensates"!) The p-wave pairing state in $^3He$ leads to the question of whether or not this state is adopted by any superconductors? One is tempted to look at metals that have strong spin-spin interactions between the electrons, hence they may show tendencies for ferromagnetism or anti-ferromagnetism at low temperatures. Somehow the material has to strike a balance between long-range magnetism and a coherent superconducting state.

It is important to define what is meant by the phrase 'unconventional superconductor'. This gets into the symmetry properties and group theoretical description of crystalline solids. Think of the superconducting gap as a function of location on the Fermi surface, $\Delta_{\vec{k}}$. Up to this point we have considered only isotropic gaps: $\Delta_{\vec{k}} = \Delta$. A conventional superconductor is one in which the gap is invariant under all symmetry operations that leave the lattice invariant, in other words $\Delta_{\hat{R}\vec{k}} = \Delta_{\vec{k}}$, where $\hat{R}$ is a symmetry operation of the lattice. A superconductor is said to be unconventional if $\Delta_{\hat{R}\vec{k}} \neq \Delta_{\vec{k}}$ for at least one symmetry operation $\hat{R}$.

The order parameter in $^3He$ was expanded in terms of spherical harmonics on the spherical Fermi surface. In the case of a crystalline solid, the symmetry is discrete, and one has to identify the irreducible representations ('Irreps', $\Gamma$) of the full symmetry group of the crystal. The dependence of the gap on direction in k-space is $\Delta_{\vec{k}} = \sum_{\Gamma m}^{d} \eta_{\Gamma m} f_{\Gamma m}(\vec{k})$, where the functions $f_{\Gamma m}(\vec{k})$ are a complete set of basis functions for the given irreducible representation of dimension $d$. For a conventional superconductor the relevant Irrep is of dimension $d = 1$ and consists of just the identity operation: $\Gamma =$ '$E$' (where E stands for the identity element, meaning that the object is left unchanged), giving rise to a complex scalar order parameter. In a spin-triplet superconductor one has to consider symmetries of the vector order parameter $d(\vec{k})$, as well as the lattice. In this case one has $d_\nu(\vec{k}) = \sum_{\Gamma m}^{3\times d} \eta_{\Gamma m,\nu} f_{\Gamma m,\nu}(\vec{k})$. Clearly there can be multiple components for the order parameter. More detailed discussion can be found in Annett's textbook and V.P. Mineev, K. Samokhin, *Introduction to Unconventional Superconductivity.* (Taylor & Francis, 1999).

Experimentally, $UPt_3$ is the clearest example of an unconventional superconductor with multiple order parameters. As shown on the class web site, this material consistently shows two nearby superconducting phase transitions through the specific heat measurements as a function of temperature. This is consistent with having multiple terms in the sum on $\Gamma m$ above, and the differences in $T_c$ values is due to a perturbation that breaks a degeneracy between these two states. As shown on the class web site, the two transitions evolve systematically in a dc magnetic field, consistent with the existence of a vector order parameter $\vec{d}$.

There is a class of heavy Fermion superconductors that show strong Fermi liquid mass renormalization in the normal state, with $m^* >> m_e$. These materials generally have un-occupied f-electron states that give rise to strong electron-electron interactions, which may then play a role in the pairing mechanism. These materials are also generally close to magnetic phase transitions, and thus may condense into unconventional superconducting states.